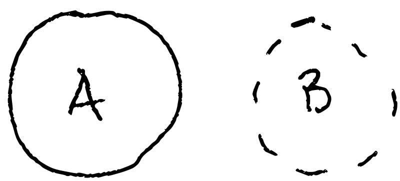
Lecture 3

- Thm 1 + proof from Lecture 2 notes.

Def ① $A \subseteq X$ is a <u>component</u> if A is a maximal connected subset, i.e. if $A \subseteq B$ and B connected $\Rightarrow A = B$.

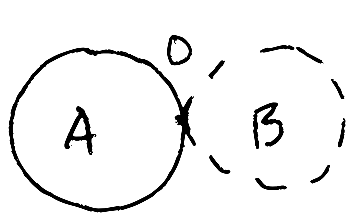
Ex ① Consider $X \subseteq \mathbb{C}$ consisting of a closed disk A and an open disk B.

Case I



$A \cap \overline{B} = \emptyset$

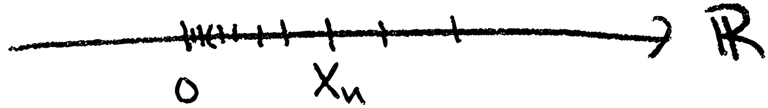
In this case A is both open and closed (in X!) since $\forall a \in A \ \exists \ B_X(a, \varepsilon) = \{x \in X : |x-a| < \varepsilon\} \subseteq A$. B is also open (even in $\mathbb{C}$). Thus, both A, B are also closed. Hence, X is not connected. A and B are connected (cf. Thm 1) and are components.

Case II. 

$A \cap B = \emptyset$ but $A \cap \overline{B} = \{0\}$.

In this case, A is no longer open. X is connected and hence the only component.

(2) Consider $X \subseteq \mathbb{R}$, $X = \{0\} \cup \{x_n \downarrow 0\}$



Each $\{x_n\}$ is open and closed and connected $\Rightarrow$ $\{x_n\}$ is a component. How about $\{0\}$? $\{0\}$ is not open, but is closed. Being a singleton, it is connected, but $0 \in \overline{\bigcup_{n=1}^{\infty} \{x_n\}}$. $\{0\}$ is a component but not obviously so at this point. Follows from Thm 2 below.

Thm2 Let (X,d) be metric space.

(i) Every $x_0 \in X$ belongs to some component.

(ii) If $C_1, C_2 \subseteq X$ are components,

$C_1 \cap C_2 = \emptyset$.

(iii) If $C \subseteq X$ is a component, then C is closed.

Pf. Can be completed (DIY) by using

Lemma1. Suppose $x_0 \in X$ and $x_0 \in A_i \subseteq X$, for A_i, i in some index set I, and all A_i are connected. Then $A = \bigcup_{i \in A} A_i$ is connected.

Pf. Suppose $B \subseteq \bigcup_i A_i$ is both open and closed. Then, $B_i = A_i \cap B$ is open and closed in A_i. A_i connected $\Rightarrow$ either $B_i = A_i$ or $B_i = \emptyset$. But $B_i = \emptyset \Rightarrow x_0 \notin B_i$ and then $B_i = \emptyset$ $\forall i$ since $x_0 \in A_i$ $\forall i$.

Thus, either all $B_i = A_i$ or all $B_i = \emptyset$.

$\Rightarrow$ either $B = \bigcup_i B_i = \bigcup_i A_i$ or $B = \emptyset$.

$\blacksquare$

<u>Rem</u>. Use Thm 2 to deduce $\{0\}$ is a component in Ex (2).

<u>Thm 3</u>. Let $G \subseteq \mathbb{C}$ be open. Then the components of G are open and at most countably many.

<u>Pf</u>. Let $C \subseteq G$ be a component. Let $a \in C$. Since G is open, $\exists\, B(a,\varepsilon) \subseteq G$. If $B(a,\varepsilon) \nsubseteq C$, then $C \cup B(a,\varepsilon)$ would be connected, by Lemma 1, and $C \subsetneq C \cup B(a,\varepsilon)$, which contradicts maximality.

Now, since each component is open, it contains a rational cplx nbr and each such belongs to at most one component. $\Rightarrow$ countably many. $\blacksquare$